Dimensions for skein module of Dehn fillings along 2-bridge knots

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New perspectives on skein modules

For M a compact oriented 3-manifold, let $S(M, \mathbb{Q}[A^{\pm 1}])$ its Kauffman bracket skein module over $\mathbb{Q}[A^{\pm 1}]$ and denote by $S(M)$ (resp. $S_\zeta(M)$) its localisation $S(M, \mathbb{Q}[A^{\pm 1}]) \otimes \mathbb{Q}(A)$ (resp. $S(M, \mathbb{Q}[A^{\pm 1}]) \underset{A=\zeta}{\otimes} \mathbb{C}$). If M is closed, an essential result of [GJS23] states that its Kauffman bracket skein module $S(M)$ over $\mathbb{Q}(A)$ is of finite dimension. However the dimension itself is not known in general.
This poster is a presentation of a work in progress that aims to compute the dimension of $S(M)$ when M is **the Dehn filling** $E_K(r)$ of slope $r \in \mathbb{Q}$ along a two bridge knot K. Denote by E_K the exterior of K.

The character variety

By a result of [PS00], it is known that the coordinate ring of the $SL_2(\mathbb{C})$-character scheme of M,

$$\chi(M) = Hom(\pi_1(M), SL_2(\mathbb{C})) /\!\!/ SL_2(\mathbb{C})$$

is isomorphic to $S_{-1}(M)$.
On the other hand, it is shown in [BW16] and [Lê15] that for ζ a primitive $2N$ root of unity with N odd, $S_\zeta(M)$ has a structure of $S_{-1}(M)$-module (equivalently $\mathbb{C}[\chi(M)]$-module).

The statement

Work In Progress 1. *Let K be a two bridge knot, for almost all slopes $r \in \mathbb{Q}$,*

$$dim_{\mathbb{Q}(A)} S(E_K(r)) = |\chi(E_K(r))|$$

How it works

For a good choice of ζ, $S(E_K(r))$ and $S_\zeta(E_K(r))$ have the same dimension

It is proved in [Lê06] that the skein module $S(E_K, \mathbb{Q}[A^{\pm 1}])$ of the exterior of the two bridge knot K is finitely generated over $\mathbb{Q}[A^{\pm 1}][m]$ where m is a meridian of K. By adapting some arguments of [Det21], we claim that, for almost all r, it implies the existence of a polynomial U such that $S(E_K(r), \mathbb{Q}[A^{\pm 1}]) \otimes \mathbb{Q}[A^{\pm 1}][\frac{1}{U}]$ is finitely generated. Then, a similar argument as the one used in [DKS25] is the following : since $R_U := \mathbb{Q}[A^{\pm 1}][\frac{1}{U}]$ is principal, the fact that the localisation $S(E_K(r), R_U) = S(E_K(r), \mathbb{Q}[A^{\pm 1}]) \otimes R_U$ is finitely generated gives the following decomposition :

$$S(E_K(r), R_U) = F \bigoplus_i R_U / q_i^{s_i}$$

where F is a free module over R_U and the direct sum is over certain powers of certain monic irreducible polynomials $q_i \in R_U$, $q_i \neq 1$, possibly repeating themselves. It is important to notice here that F has the same dimension as $S(E_K(r))$.
Choosing a primitive ($2N$, N odd-)root of unity ζ such that it is not a root of any q_i not U gets the following :

$$S_\zeta(E_K(r)) = S(E_K(r), R_U) \underset{A=\zeta}{\otimes} \mathbb{C} = \mathbb{C}^{\dim F}$$

And $S_\zeta(E_K(r))$ would then be of the same dimension as $S(E_K(r))$.

$S_\zeta(E_K(r))$ is isomorphic to $\mathbb{C}[\chi(E_K(r))]$

The latest also implies that $\chi(E_K(r))$ is finite. Because of the work done in [FKBL25], [FTFKB25] and [Kor25], we expect the decompositions of $\mathbb{C}[\chi(E_K(r))]$ and $S_\zeta(E_K(r))$ into its localisations (as $\mathbb{C}[\chi(E_K(r))]$-module) at the maximal ideals of $\mathbb{C}[\chi(E_K(r))]$ to match and get that $\mathbb{C}[\chi(E_K(r))] \simeq S_\zeta(E_K(r))$.

To go further

By Culler-Shallen theory [CS83], 3-manifolds with infinite $\chi(M)$ contain incompressible surfaces but the converse is not true. However, with a good definition of tameness for $S(M, \mathbb{Q}[A^{\pm 1}])$, having infinite $\chi(M)$ implies that $S(M, \mathbb{Q}[A^{\pm 1}])$ is not tame and it is conjectured in [DKS25] that

Conjecture 1. *If M is closed and does not contain incompressible surfaces, then $S(M, \mathbb{Q}[A^{\pm 1}]))$ is finitely generated.*

A good exemple where this conjecture could be a refinement of the character variety in term of detection of incompressible surfaces is the Dehn filling of slope 4 along the figure eight knot, where the manifold contains an incompressible torus but has finite character variety.

Work In Progress 2. *Computing the skein module of the Dehn filling of slope 4 along the figure eight knot.*

Among other things, we expect that $S(M)$ does not have $|\chi(M)|$ as dimension and therefore (by a result of [DKS25]) $S(M, \mathbb{Q}[A^{\pm 1}])$ should not be finitely generated.

Bibliography

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