DIMENSION OF THE SKEIN MODULE OF A DEHN FILLING

EDWIN KITAEFF

ABSTRACT. Given a knot K and a generic slope r, we study the Kauffman bracket skein module (KBSM) $S(E_K(r), \mathbb{Q}(A))$ of the Dehn filling $E_K(r)$ of slope r along K, assuming that the KBSM $S(E_K, \mathbb{Q}[A^{\pm 1}])$ of the exterior E_K of K is finitely generated over $S(\partial E_K, \mathbb{Q}[A^{\pm 1}])$. As shown in [Lê06], this condition is satisfied for K a two-bridge knot. In this setting, we show that $\dim_{\mathbb{C}}(S_\zeta(E_K(r))) = \dim_{\mathbb{Q}(A)}(S(E_K(r)))$ for almost all primitive $2N$-th roots of unity ζ with N odd, and for almost all slopes r. When the character variety of a 3-manifold M is finite (which is the case for these Dehn fillings), we also discuss the decomposition of $S_\zeta(M)$ in terms of localized skein modules. In particular, the dimension of the localized skein modules at a non-central point is the multiplicity of this point in the character variety.

1. INTRODUCTION

1.1. The Kauffman bracket skein module.

Let M be a compact oriented 3-manifold, let R be a commutative ring and let A be a choice of an invertible element of R. The Kauffman bracket skein module $S_A(M, R)$, or simply skein module here, was introduced independently by Przytycki ([Prz91]) and Turaev ([Tur88]). It is defined as the R-module spanned by the framed links in M modulo isotopies and the Kauffman skein relations:

For a surface Σ, we write $S_A(\Sigma, R)$ instead of $S_A(\Sigma \times I, R)$.

If the choice of A is clear, we simply write $S(M, R)$, or even $S(M)$ if $R = \mathbb{Q}(A)$. For $\zeta \in \mathbb{C}^*$, we define $S_\zeta(M) := S_\zeta(M, \mathbb{C})$.

Although the definition of the skein module is simple, its computation is notoriously difficult. In fact, the skein module $S(M, \mathbb{Q}[A^{\pm 1}])$ is known only for a limited number of 3-manifolds, such as lens spaces [HP93, Theorem 4], $\mathbb{S}^2 \times \mathbb{S}^1$[HP95], the exterior of a 2-bridge knot [Lê06, Theorem 2], $\mathbb{R}P^3 \# \mathbb{R}P^3$ [Mro11, Theorem 1], and $(\mathbb{S}^2 \times \mathbb{S}^1) \# (\mathbb{S}^2 \times \mathbb{S}^1)$ [BKSW25, §3].

It is worth noting that the structure of $S(M, \mathbb{Q}[A^{\pm 1}])$ can be complicated as $S(\mathbb{S}^2 \times \mathbb{S}^1, \mathbb{Q}[A^{\pm 1}])$ is not finitely generated and $S(\mathbb{R}P^3 \# \mathbb{R}P^3, \mathbb{Q}[A^{\pm 1}])$ is not even a sum of finitely generated modules. However, when $R = \mathbb{Q}(A)$ we get a simpler structure for the skein module. Mainly, we know the following striking result:

Theorem 1.1. [GJS23, Theorem 1] *For M a closed 3-manifold, $S(M)$ is a finite dimensional $\mathbb{Q}(A)$-vector space.*

Unfortunately, the proof of [GJS23] is not constructive and cannot be used to compute $S(M)$. An alternative proof can be found in [BD25], where, unlike [GJS23], the dimension of $S(M)$ is bounded (from above) using an algorithm that computes an explicit set of generators. However, the algorithm is computation-heavy and the result set is often not optimal.

We also know a handful more examples of closed 3-manifolds than for $R = \mathbb{Q}[A^{\pm 1}]$ in the case $R = \mathbb{Q}(A)$, most notably $\Sigma \times \mathbb{S}^1$ [GM19, DW21], most surgeries on the figure eight knot [DKS25a], and small Seifert manifolds [DKS25b].

On the other hand, the interpretation of the dimension of $S(M)$ is the subject of the following conjecture:

Conjecture 1.2. [GS24, Conjecture C] *For a closed 3-manifold M, we have that*

$$\dim_{\mathbb{Q}(A)}(S(M)) = \dim_{\mathbb{C}} HP^0_{\#}(M)$$

where $HP_{\#}(M)$ is the Abouzaid-Manolescu homology [AM20].

Overall, the computation and interpretation of the dimension of $S(M)$ remain a very difficult and open problem.

1.2. **The results.** This work extends a Theorem of [DKS25a] which was used to compute the skein module of small Seifert manifolds in [DKS25b]. We recall their result below.

For an oriented connected manifold M, let

$$\chi(M) = \text{Hom}(\pi_1(M), SL_2(\mathbb{C})) /\!\!/ SL_2(\mathbb{C})$$

be the $SL_2(\mathbb{C})$-character scheme of M and $X(M)$ its underlying algebraic set. The character variety $X(M)$ can be seen as the quotient of $Hom(\pi_1(M), SL_2(\mathbb{C}))$ in which two representations are identified if and only if their traces coincide. If $\mathbb{C}[\chi(M)]$ has no non-trivial nilpotent elements, we say that the character variety is reduced. It is worth noting that, when $X(M)$ is finite and reduced, Conjecture 1.2 becomes that the dimension of $S(M)$ is the number of characters of $X(M)$ [AM20, Theorem 1.4].

It is well known that, by [Bul97, PS00, Lê15, BW16], $S_\zeta(M)$ has a structure of $\mathbb{C}[\chi(M)]$-module (see Section 2 for more details).

The module $S(M, \mathbb{Q}[A^{\pm 1}])$ is called tame if it can be expressed as a direct sum of cyclic $\mathbb{Q}[A^{\pm 1}]$-modules and does not contain $\mathbb{Q}[A^{\pm 1}]/(\phi_{2N})$ as a submodule for at least one odd N, where ϕ_{2N} is the $2N$-th cyclotomic polynomial.

Their result is then the following:

Theorem 1.3. [DKS25a, Theorem 1.1] *Let M be a closed 3-manifold such that $S(M, \mathbb{Q}[A^{\pm 1}])$ is tame and $X(M)$ is finite.*
Then, for almost all primitive $2N$-th roots of unity ζ with N odd,

$$\dim_{\mathbb{Q}(A)} S(M) = \dim_{\mathbb{C}} S_\zeta(M) = |X(M)|$$

where $|X(M)|$ is the number of points in $X(M)$ counted with multiplicity.

Remark 1.4. *The original version of this theorem required $X(M)$ to be reduced. However, one can use the work of [FTFKB25] to remove this condition. We make this claim more precise in Section 4.5.*

Unfortunately, the tameness condition is not easy to check. The closed 3-manifolds known to be tame typically satisfy the stronger condition of having $S(M, \mathbb{Q}[A^{\pm 1}])$ finitely generated over $\mathbb{Q}[A^{\pm 1}]$ and are essentially Dehn fillings on the figure-eight knot [DKS25a, Theorem 4.3] and small Seifert manifolds [DKS25b, Theorem 1.2]. Nonetheless, it is conjectured in [DKS25b, Conjecture 1.1] that every closed small 3-manifold is tame.

In Section 3.2, we will prove the first equality of Theorem 1.3 for a family of Dehn fillings including Dehn fillings on two-bridge knots without requiring the tameness condition.

Recall that for a knot K with exterior E_K, as a skein module of a manifold with torus-boundary, the skein module $S(E_K, \mathbb{Q}[A^{\pm 1}])$ has the structure of an $S(\mathbb{T}^2, \mathbb{Q}[A^{\pm 1}])$-module, then:

Theorem 1.5 (Main Theorem 1). *Let K be a knot such that $S(E_K, \mathbb{Q}[A^{\pm 1}])$ is a finitely generated $S(\partial E_K, \mathbb{Q}[A^{\pm 1}])$-module. Then, for almost all slopes r and almost all primitive $2N$-th roots of unity ζ,*

$$\dim_{\mathbb{Q}(A)} S(E_K(r)) = \dim_{\mathbb{C}} S_{\zeta}(E_K(r))$$

Where $E_K(r)$ is the Dehn filling of slope r along K.

Remark 1.6. *Note that because of* [DKS25a, Theorem 2.1], *for infinitely many ζ, $\dim_{\mathbb{C}}(S_{\zeta}(E_K(r)))$ is greater than or equal to the number of points in $X(E_K(r))$ (without multiplicity) and thus $X(E_K(r))$ is finite.*

Remark 1.7. *According to* [Lê06, Theorem 2], *if K is a 2-bridge knot, $S(E_K, \mathbb{Q}[A^{\pm 1}])$ is finitely generated over $S(\partial E_K, \mathbb{Q}[A^{\pm 1}])$. Therefore the result of Theorem 1.5 is valid for almost all Dehn fillings on two-bridge knots.*

A small knot is a knot for which the exterior does not contain any essential surface. Generically, a knot that is not small does not have $S(E_K, \mathbb{Q}[A^{\pm 1}])$ finitely generated over $S(\partial E_K, \mathbb{Q}[A^{\pm 1}])$. By contrast, it is widely believed by experts that this condition is satisfied for small knots.

Remark 1.8. *Having $S(E_K, \mathbb{Q}[A^{\pm 1}])$ finitely generated over $S(\partial E_K, \mathbb{Q}[A^{\pm 1}])$ still makes sense when E_K is replaced by a 3-manifold M with $\partial M \simeq \mathbb{T}^2$ and our main result (Theorem 1.5) applies to that setting as well.*

A character $[\rho] \in X(M)$ is said irreducible if ρ is irreducible, central if $Im(tr\ \rho) = \{\pm 2\}$ and abelian non-central otherwise. Moreover, the notion of reduced skein module, introduced in [FKBL19], associates to a character $[\rho]$ a quotient of $S_{\zeta}(M)$ denoted $S_{\zeta, [\rho]}(M)$. The reduced skein modules, along with the localized skein modules, will be introduced in more details in Section 2.

In the following theorem, we give a replacement for the second equality of Theorem 1.3 under a condition including the Dehn fillings we considered in Theorem 1.5:

Theorem 1.9 (Main Theorem 2). *Let M be a closed oriented 3-manifold such that $\chi(M)$ is finite. For $[\rho] \in \chi(M)$, let $n_{[\rho]}$ be the multiplicity of $[\rho]$. Let $\chi_0 \subset \chi(M)$ be the set of central characters and let $\mathbb{1}$ be the trivial representation.*
Then, for all primitive $2N$-th roots of unity ζ with N odd,

$$S_{\zeta}(M) = (S_{\zeta, [\mathbb{1}]}(M))^{|\chi_0|} \bigoplus_{[\rho] \in \chi(M) \setminus \chi_0} \mathbb{C}^{n_{[\rho]}}$$

According to [DKS25a, Theorem 2.2], for any $[\rho] \in X(M)$, $\dim_{\mathbb{C}} S_{\zeta, [\rho]}(M) \geq 1$. Then we have the following corollary:

Corollary 1.10. *Let M be a closed oriented 3-manifold such that $\chi(M)$ is finite, then:*

$$\dim_{\mathbb{C}} S_{\zeta}(M) \geq |X(M)|$$

Where $|X(M)|$ is counted with multiplicity.

By Remark 1.6, when $S(E_K, \mathbb{Q}[A^{\pm 1}])$ is finitely generated over $S(\partial E_K, \mathbb{Q}[A^{\pm 1}])$, Theorem 1.9 applies to $E_K(r)$ for almost all slopes r, therefore we get:

Corollary 1.11. *Let K be a knot such that $S(E_K, \mathbb{Q}[A^{\pm 1}])$ is finitely generated over $S(\partial E_K, \mathbb{Q}[A^{\pm 1}])$ and ζ be a primitive $2N$-th root of unity with N odd.*
Then for almost all slopes $r \in \mathbb{Q}$, $\dim_{\mathbb{Q}(A)} S(E_K(r)) \geq |X(E_K(r))|$, where $|X(E_K(r))|$ is counted with multiplicity.

Unfortunately, we do not know how to compute $S_{\zeta, [\mathbb{1}]}(M)$ yet, nonetheless we propose the following conjecture:

Conjecture 1.12. *Let M be an oriented closed 3-manifold and ζ be a primitive $2N$-th root of unity with N odd. If $X(M)$ is finite, then*

$$S_{\zeta,[\mathbb{1}]}(M) \simeq \mathbb{C}$$

If this conjecture is true, a knot K such that $S(E_K, \mathbb{Q}[A^{\pm 1}])$ is finitely generated over $S(\partial E_K, \mathbb{Q}[A^{\pm 1}])$ will also verify that, for almost all $r \in \mathbb{Q} \cup \{\infty\}$:

$$\dim_{\mathbb{Q}(A)} S(E_K(r)) = \dim_{\mathbb{C}} S_\zeta(E_K(r)) = |X(E_K(r))|$$

Where $|X(E_K(r))|$ is the number of characters of $X(E_K(r))$ counted with multiplicity.

We first introduce the fundamental notions connecting the character variety to the skein modules in Section 2. Then, we prove Theorem 1.5 in Section 3. To do so, we adapt a proof of [Det21] to find a finitely generated localization of $S(E_K(r), \mathbb{Q}[A^{\pm 1}])$. After which, we follow a line of reasoning presented in [DKS25a] to show that the free part of this localization has the same rank as $S(E_K(r))$ and $S_\zeta(E_K(r))$ for almost all roots of unity ζ of order $ord(\zeta) \equiv 2$ (mod 4).

In Section 4 we adapt the proofs of [FTFKB25] to non-central characters in order to show Theorem 1.9.

2. NOTATION AND PRELIMINARIES

We recall the relation between character variety and skein module below.

The first fundamental result is connecting $S_{-1}(M)$ with the character variety by:

Theorem 2.1. [Bul97, PS00] *The following linear map is an isomorphism :*

$$\tau : S_{-1}(M) \longrightarrow \mathbb{C}[\chi(M)]$$

$$L \longmapsto \prod_{i=1}^{n}(-tr(K_i))$$

Where $K_1, \ldots, K_n$ are the components of the link L.

It was initially established in [Bul97] up to nilpotents, and later completely proven in [PS00].

The structure of $S_\zeta(M)$ is also deeply connected to the character variety through the threading map of [BW16] which we describe below. First, we need to recall the definition of Chebychev polynomials of the first kind:

$$(\text{T}) \qquad \begin{cases} T_0 = 2, \ T_1 = X \\ \forall n \geq 2, \ T_n = X T_{n-1} - T_{n-2} \end{cases}$$

Let ζ be a primitive $2N$-th root of unity with N odd. As demonstrated in [Lê15], for a link L, the element $T_N(L) \sqcup L' \in S_\zeta(M)$ depends solely on L' and the homotopy class of $T_N(L)$. Let $\tau : S_{-1}(M) \to S_\zeta(M)$ be the linear map defined by $\tau(L) = T_N(L)$.

Then,

Theorem 2.2. [BW16][Lê15] *For ζ a primitive $2N$-th root of unity with N odd, τ gives $S_\zeta(M)$ the structure of a $S_{-1}(M)$-module.*

Then, thanks to Bullock-Przytycki-Sikora's Theorem, $S_\zeta(M)$ has a structure of $\mathbb{C}[\chi(M)]$-module.

As an affine variety, the maximal ideals of $\mathbb{C}[\chi(M)]$ correspond to the points of $\chi(M)$, we denote $MaxSpec(S_{-1}(M)) = \{\mathfrak{m}_{[\rho]}, [\rho] \in \chi(M)\}$.

Following [FTFKB25], we define the reduced skein module at a character $[\rho] \in \chi(M)$ to be:

$$S_{\zeta,[\rho]}(M) := S_\zeta(M) \bigotimes_{S_{-1}(M)} \left. S_{-1}(M) \middle/ \mathfrak{m}_{[\rho]} \right.$$

Similarly, the localized skein module at $[\rho]$ is:

$$S_\zeta(M)_{[\rho]} := S_\zeta(M) \bigotimes_{S_{-1}(M)} (S_{-1}(M) \setminus \mathfrak{m}_{[\rho]})^{-1} S_{-1}(M)$$

When $[\rho]$ is an isolated and reduced point of $S_{-1}(M)$, we have that $S_{-1}(M)_{[\rho]} \simeq S_{-1,[\rho]}(M) \simeq \mathbb{C}$ and the localized skein module at $[\rho]$ has the same dimension over $\mathbb{C}$ as the reduced skein module at $[\rho]$.

Let L, L' be links in M and $K_1, \ldots, K_n$ be the components of L. Because of the structures given by Theorem 2.1 and Theorem 2.2, the following relation holds in $S_{\zeta,[\rho]}(M)$:

$$T_N(L) \sqcup L' = (\prod_{i=1}^{n} -tr(\rho(K_i)))L \in S_{\zeta,[\rho]}(M)$$

In fact $S_{\zeta,[\rho]}(M)$ is the quotient of $S_\zeta(M)$ by all relations of this type.

When $\chi(M)$ is finite, $S_{-1}(M)$ is Artinian, we then have the following decomposition:

$$S_\zeta(M) = \bigoplus_{[\rho] \in \chi(M)} S_\zeta(M)_{[\rho]}$$

3. Stabilization of the dimension of $S_\zeta(E_K(r))$ at $\dim_{\mathbb{Q}(A)} S(E_K(r))$

3.1. **A finitely generated localization of $S(E_K(r), \mathbb{Q}[A^{\pm 1}])$.** For a polynomial $U \in \mathbb{Q}[A^{\pm 1}]$, denote $R_U := \mathbb{Q}[A^{\pm 1}][U^{-1}]$.

The main result of this section will be Proposition 3.5. Nonetheless, it needs some technicalities to be stated in its precise form. For clarity, we give a paraphrase here:

Proposition 3.1. *Let K be a knot such that $S(E_K, \mathbb{Q}[A^{\pm 1}])$ is a finitely generated $S(\partial E_K, \mathbb{Q}[A^{\pm 1}])$-module, then there exists a polynomial U such that, for almost all slopes r, $S(E_K(r), R_U)$ is finitely generated over R_U.*

The main tool here is the Frohman-Gelca basis of $S(\mathbb{T}^2, \mathbb{Q}[A^{\pm 1}])$ used on $S(E_K, \mathbb{Q}[A^{\pm 1}])$ through its $S(\partial E_K, \mathbb{Q}[A^{\pm 1}]) \simeq S(\mathbb{T}^2, \mathbb{Q}[A^{\pm 1}])$-module structure. We describe the Frohman-Gelca basis below.

Fixing two oriented curves λ and μ intersecting once on $\mathbb{T}^2$, let x, y be coprime integers, we define $\gamma_{(x,y)}$ to be the skein element represented by an oriented curve of homology class $x\lambda + y\mu$ on $\mathbb{T}^2 \times I$. In our context, we choose λ to be a meridian of K and μ a longitude. The multicurves $\gamma_{(x,y)}^n$, consisting of n parallel copies of $\gamma_{(x,y)}$, together with the empty curve, form a basis of $S(\mathbb{T}^2, \mathbb{Q}[A^{\pm 1}])$.

Recall that the definition of Chebychev polynomials of the first kind $\{T_n\}$ is given at (T). Frohman and Gelca introduced the following basis of $S(\mathbb{T}^2, \mathbb{Q}[A^{\pm 1}])$, for which the product (stacking operation) satisfies the so-called product-to-sum formula:

Theorem 3.2. [FG00, Theorem 1] *The family $\{(x, y)_T := T_d(\gamma_{(\frac{x}{d}, \frac{y}{d})}), \ d = \gcd(x, y)\}$ is a basis for $S(\mathbb{T}^2, \mathbb{Q}[A^{\pm 1}])$ for which we have the following:*

$$(x, y)_T \star (z, t)_T = A^{xt - yz}(x + z, y + t)_T + A^{yz - xt}(x - z, y - t)_T$$

Remark 3.3. *Here, we choose the convention $(0,0)_T = 2 \cdot \emptyset$.*

We precede the proof of Proposition 3.5 with the following lemma:

Lemma 3.4. *For any knot K and for every $f \in S(E_K, \mathbb{Q}[A^{\pm 1}])$, there exists a polygon $\mathcal{P}^f$ with vertix set $S_f \subset \mathbb{Z}^2$, and coefficients $c^f_{\alpha,\beta} \in \mathbb{Q}[A^{\pm 1}]$ such that,*

$$\left(\sum_{(\alpha,\beta) \in \mathcal{P}^f \cap \mathbb{Z}^2} c^f_{\alpha,\beta} (\alpha,\beta)_T \right) \cdot f = 0$$

Where $(\alpha,\beta)_T \in S(\partial E_K, \mathbb{Q}[A^{\pm 1}]) \simeq S(\mathbb{T}^2, \mathbb{Q}[A^{\pm 1}])$.

Moreover, the coefficients can be chosen so that $\begin{cases} (-\mathcal{P}^f) = \mathcal{P}^f \\ \forall (\alpha,\beta) \in \mathcal{P}^f \cap \mathbb{Z}^2, \ c^f_{\alpha,\beta} = c^f_{-\alpha,-\beta} \\ \forall (\alpha,\beta) \in S_f, \ c^f_{\alpha,\beta} \neq 0 \end{cases}$.

Proof. It is stated in [BD25, Corollary 1.7] that as long as the boundary of a compact oriented 3-manifold M is not a disjoint union of spheres, we have that for every $f \in S(M, \mathbb{Z}[A^{\pm 1}])$, there exists a non-zero element $z \in S(\partial M, \mathbb{Z}[A^{\pm 1}])$ such that $z \cdot f = 0$. It implies in our case the existence of a non-zero element z in $S(\partial E_K, \mathbb{Q}[A^{\pm 1}])$ such that $z \cdot f = 0$.
Since $S(\partial E_K, \mathbb{Q}[A^{\pm 1}]) \simeq S(\mathbb{T}^2, \mathbb{Q}[A^{\pm 1}])$, we can express z in the Frohman-Gelca basis and get:

$$\left(\sum_{(\alpha,\beta) \in \mathcal{P}^f \cap \mathbb{Z}^2} c^f_{\alpha,\beta} (\alpha,\beta)_T \right) \cdot f = 0$$

where $\mathcal{P}^f$ is a polygon with vertices in $\mathbb{Z}^2$. Because $(\alpha,\beta)_T = (-\alpha,-\beta)_T$ in the Frohman-Gelca basis, this relation can be chosen such that $(-\mathcal{P}^f) = \mathcal{P}^f$ and $c^f_{\alpha,\beta} = c^f_{-\alpha,-\beta} \in \mathbb{Q}[A^{\pm 1}]$ for $(\alpha,\beta) \in \mathcal{P}^f \cap \mathbb{Z}^2$. Moreover, $c^f_{\alpha,\beta} \neq 0$ for $(\alpha,\beta) \in \partial \mathcal{P}^f \cap \mathbb{Z}^2$. $\qquad \square$

Let K be a knot such that there exists a set of generators F of $S(E_K, \mathbb{Q}[A^{\pm 1}])$ over $S(\partial E_K, \mathbb{Q}[A^{\pm 1}])$. For each $f \in F$, let $\mathcal{P}^f$ and $c^f_{\alpha,\beta}$ be given by Lemma 3.4, denote S_f the set of vertices of $\mathcal{P}^f$ and let

$$U := \prod_{f \in F} \prod_{(\alpha,\beta) \in S_f} c^f_{\alpha,\beta}$$

We now have introduced all the elements we needed to state Proposition 3.1 more precisely:

Proposition 3.5. *For all slopes r such that r^{-1} is not a slope of any of the polygons $\mathcal{P}^f$, $S(E_K(r), R_U)$ is finitely generated over $R_U = \mathbb{Q}[A^{\pm 1}][U^{-1}]$.*

Proof. To start with, since $S(E_K, R_U) = S(E_K, \mathbb{Q}[A^{\pm 1}]) \otimes R_U$, F also generates $S(E_K, R_U)$ over $S(\partial E_K, R_U)$.
Since every element of $S(E_K(r), R_U)$ can be isotoped into E_K, to show that $S(E_K(r), R_U)$ is finitely generated over R_U, it suffices to show that $S(E_K, R_U)$ is finitely generated over R_U as a subspace of $S(E_K(r), R_U)$. This can be done by showing that $S(\partial E_K, R_U) \cdot f \subset S(E_K(r), R_U)$ is finitely generated over R_U for every $f \in F$. In the following, we fix a generator $f \in F$.

First, we can multiply the relation of Lemma 3.4 on the left with an element $(\mu, \nu)_T \in S(\partial E_K, R_U)$. Then, using the product-to-sum formula, we obtain:

$$0 = (\mu, \nu)_T \star \left(\sum_{(\alpha, \beta) \in \mathcal{P}^f \cap \mathbb{Z}^2} c_{\alpha, \beta}^f (\alpha, \beta)_T \right) \cdot f$$

$$= \left(\sum_{(\alpha, \beta) \in \mathcal{P}^f \cap \mathbb{Z}^2} A^{\mu\beta - \nu\alpha} c_{\alpha, \beta}^f (\alpha + \mu, \beta + \nu)_T + \sum_{(\alpha, \beta) \in \mathcal{P}^f \cap \mathbb{Z}^2} A^{-\mu\beta + \nu\alpha} c_{\alpha, \beta}^f (\alpha - \mu, \beta - \nu)_T \right) \cdot f$$

$$= \left(\sum_{(\alpha, \beta) \in \mathcal{P}^f \cap \mathbb{Z}^2} A^{\mu\beta - \nu\alpha} c_{\alpha, \beta}^f (\alpha + \mu, \beta + \nu)_T + \sum_{(\alpha, \beta) \in (-\mathcal{P}^f) \cap \mathbb{Z}^2} A^{\mu\beta - \nu\alpha} c_{-\alpha, -\beta}^f (-\alpha - \mu, -\beta - \nu)_T \right) \cdot f$$

Since $(-\alpha - \mu, -\beta - \nu)_T = (\alpha + \mu, \beta + \nu)_T$, $c_{-\alpha, -\beta}^f = c_{\alpha, \beta}^f$ and $(-\mathcal{P}^f) = \mathcal{P}^f$, the last line becomes:

$$(1) \qquad \left(\sum_{(\alpha, \beta) \in \mathcal{P}^f \cap \mathbb{Z}^2} 2 A^{\mu\beta - \nu\alpha} c_{\alpha, \beta}^f (\alpha + \mu, \beta + \nu)_T \right) \cdot f = 0$$

Keeping in mind Relation (1), we get a second relation from the surgery : performing a surgery of slope $r = \dfrac{p}{q}$ (with $\gcd(p, q) = 1$) on K makes the curve $\gamma_{p,q} = (p, q)_T$ trivial. Thus, $(p, q)_T \cdot f = (-A^2 - A^{-2}) \cdot f$ in $S(E_K(r), R_U)$. We then multiply by $(\mu, \nu)_T$ on the right and use the product-to-sum formula to deduce more relations, we obtain:

$$(2) \qquad \left(A^{p\nu - q\mu} (p + \mu, q + \nu)_T + (-A^2 - A^{-2})(\mu, \nu)_T + A^{q\mu - p\nu} (p - \mu, q - \nu)_T \right) \cdot f = 0$$

To show that $S(\partial E_K, R_U) \cdot f$ is finitely generated over R_U, choose two morphisms $\lambda, \epsilon : \mathbb{Z}^2 \to \mathbb{Z}$ such that $\lambda \neq 0$, $\lambda(p, q) = 0$ and $\epsilon(p, q) = 1$.

Since $\dfrac{q}{p}$ is not a slope of $\mathcal{P}^f$, λ has a unique maximum M and a unique minimum $-M$ on $\mathcal{P}^f$. Let (x, y) be such that $\lambda(x, y) \geq M$ and let (a, b) realize the maximum for λ over $\mathcal{P}^f$. Relation (1) with $(\mu, \nu) := (x - a, y - b)$ gets $(x, y) = (a + \mu, b + \nu)$ to be the unique maximum for λ between all the vertices involved in the relation. Since $c_{(a,b)}^f \neq 0$, this gives an expression of $(x, y)_T \cdot f$ as a linear combination of elements with lesser images by λ. Note that we need to inverse the coefficients $2 A^{\mu\beta - \nu\alpha} c_{\alpha, \beta}^f$ for every vertex (α, β) of $\mathcal{P}^f$, which may not be possible in $\mathbb{Q}[A^{\pm 1}]$ but is possible in R_U.
By doing this also for the unique minimum $-M$ of $\mathcal{P}^f$, we find that $S(\partial E_K, R_U) \cdot f$ is spanned by elements $(x, y)_T \cdot f$ such that $-M \leq \lambda(x, y) \leq M$.

Similarly, since $A^{p\nu - q\mu}$ is invertible and because $(p - \mu, q - \nu)_T = (\mu - p, \nu - q)_T$, relation (2) expresses $(\mu + p, \nu + q)_T \cdot f$ (resp. $(\mu - p, \nu - q)_T \cdot f$) as a linear combination of elements with same image by λ but lesser (resp. greater) image by ϵ.

In the end, $S(\partial E_K, R_U) \cdot f$ is spanned by elements $(x, y)_T \cdot f$ such that $-M \leq \lambda(x, y) \leq M$ and $0 \leq \epsilon(x, y) \leq 1$ which have coordinates in the intersection of two non-parallel bands of $\mathbb{Z}^2$ and thus form a finite set. $\square$

Remark 3.6. *Fixing the slope and the associated λ (if possible), the choice of U can be reduced to the product of coefficients $c_{\alpha, \beta}^f$ with (α, β) realising the maximum and the minimum of λ on $\mathcal{P}^f$ for each generator f.*

3.2. **The proof of Theorem 1.5.** We adapt the method of [DKS25a, Theorem 3.1] for a Dehn filling over a knot K such that $S(E_K, \mathbb{Q}[A^{\pm 1}])$ is finitely generated over $S(\partial E_K, \mathbb{Q}[A^{\pm 1}])$:

Proof of Theorem 1.5. By Proposition 3.5, there exists a polynomial $U \in \mathbb{Q}[A^{\pm 1}]$ for which $S(E_K(r), R_U)$ is finitely generated over $R_U = \mathbb{Q}[A^{\pm 1}][U^{-1}]$.

The ring R_U is a PID as a localization of a PID (see [AM69, Prop. 3.11] for instance). Then, $S(E_K(r), R_U)$ is finitely generated over R_U and thus, see for instance [Hun80, Theorem IV.6.12], has a decomposition as

$$S(E_K(r), R_U) = F \bigoplus_i {R_U}\big/_{q_i^{s_i}}$$

where F is a free R_U-module and the direct sum is finite over certain powers of certain irreducibles $q_i \in R_U$, $q_i \neq 1$, possibly repeating themselves.

It follows that $\dim_{\mathbb{Q}(A)}(S(E_K(r))) = rk_{R_U}(F)$:

$$S(E_K(r)) = S(E_K(r), R_U) \otimes \mathbb{Q}(A) \simeq (\mathbb{Q}(A))^{rk_{R_U}(F)}$$

On the other hand, let ζ be a primitive $2N$-th root of unity, such that ζ is not a root of any q_i or of U. Thus, ${R_U}\big/_{q_i^{s_i}} \underset{A=\zeta}{\otimes} \mathbb{C} = 0$ and:

$$S_\zeta(E_K(r)) = S(E_K(r), R_U) \underset{A=\zeta}{\otimes} \mathbb{C} \simeq \mathbb{C}^{rk_{R_U}(F)}$$

Thus, $\dim_{\mathbb{Q}(A)}(S(E_K(r))) = rk_{R_U}(F) = \dim_{\mathbb{C}}(S_\zeta(E_K(r)))$. $\qquad\square$

4. Dimension of $S_\zeta(M)$

By Remark 1.6, if K is a knot such that $S(E_K, \mathbb{Q}[A^{\pm 1}])$ is finitely generated over $S(\partial E_K, \mathbb{Q}[A^{\pm 1}])$ and for almost all slopes r, $E_K(r)$ is as a 3-manifold with finite character variety. In general, the study of the skein module $S_\zeta(M)$ of a closed oriented 3-manifold with finite character variety reduces to the study of the localized skein modules $S_\zeta(M)_{[\rho]}$ through the decomposition:

$$S_\zeta(M) = \bigoplus_{[\rho] \in \chi(M)} S_\zeta(M)_{[\rho]}$$

In order to explain the state of the art on localized skein modules, we will use some notions of affine PI algebras:

4.1. **Almost Azumaya algebras.**

Definition 4.1. *Let $\mathcal{A}$ be a $\mathbb{C}$-algebra.*
If $\mathcal{A}$ is affine, prime with finite rank over its center, then $\mathcal{A}$ is said to be almost Azumaya.

In this case (see [BG02, III.1.2]), there is an integer D such that the dimension of $\mathcal{A} \underset{Z(\mathcal{A})}{\otimes} Frac(Z(\mathcal{A}))$ over $Frac(Z(\mathcal{A}))$ is D^2. The integer D is called the PI-degree of $\mathcal{A}$.

Definition 4.2. *If $\mathcal{A}$ is almost Azumaya, the Azumaya locus is*

$$Azu(\mathcal{A}) = \{\mathfrak{m} \in MaxSpec(Z(\mathcal{A})), \mathcal{A}\big/_{\mathfrak{m}\mathcal{A}} \simeq M_D(\mathbb{C})\}$$

For a finitely generated $\mathcal{A}$-module $\mathcal{K}$, we also define

$$Azu'_{\mathcal{A}}(\mathcal{K}) := \{\mathfrak{m} \in MaxSpec(Z(\mathcal{A})), \dim_{\mathbb{C}}(\mathcal{K}\big/_{\mathfrak{m}\mathcal{K}}) = \dim_{Frac(Z(\mathcal{A}))}(\mathcal{K} \underset{Z(\mathcal{A})}{\otimes} Frac(Z(\mathcal{A})))\}$$

Proposition 4.3. [BG02, Theorem III.1.7] *$Azu(\mathcal{A})$ is Zariski open.*

Proposition 4.4. *$Azu'_{\mathcal{A}}(\mathcal{K})$ is Zariski open.*

Proof. Let $d = \dim_{Frac(Z(\mathcal{A}))}(\mathcal{K} \underset{Z(\mathcal{A})}{\otimes} Frac(Z(\mathcal{A})))$.

For $\mathfrak{p} \in Spec(Z(\mathcal{A}))$, let $\kappa(\mathfrak{p}) = {Z(\mathcal{A})_{\mathfrak{p}}} \big/ {\mathfrak{p} Z(\mathcal{A})_{\mathfrak{p}}}$ be the residue field of $\mathfrak{p}$.

Let $\Lambda : Spec(Z(\mathcal{A})) \to \mathbb{N}$ be defined by $\Lambda(\mathfrak{p}) = \dim_{\kappa(\mathfrak{p})}(\mathcal{K} \otimes \kappa(\mathfrak{p}))$.

For $\mathfrak{m} \in MaxSpec(Z(\mathcal{A}))$, we have that $\kappa(\mathfrak{m}) = \mathbb{C}$ and $\Lambda(\mathfrak{m}) = \dim_{\mathbb{C}} {\mathcal{K}} \big/ {\mathfrak{m} \mathcal{K}}$. Moreover, since $\mathcal{A}$ is prime, $Z(\mathcal{A})$ has no zero divisors, then $(0) \in Spec(Z(\mathcal{A}))$ and we have $\kappa((0)) = Frac(Z(\mathcal{A}))$ and $\Lambda((0)) = d$.

It is known that Λ is upper semi-continuous ([Har77, Example 12.7.2]). In particular, for every $\mathfrak{p} \in Spec(Z(\mathcal{A}))$, the set $\{\mathfrak{p}' \in Spec(Z(\mathcal{A})), \Lambda(\mathfrak{p}') \leq \Lambda(\mathfrak{p})\}$ is an open neighborhood of $\mathfrak{p}$.

Moreover, in the Zariski topology, (0) is included in every neighborhood of any prime ideal ((0) is called a generic point of $Spec(Z(\mathcal{A}))$).

This shows that $d = \Lambda((0))$ is the minimal dimension for $\{\kappa(\mathfrak{p})\}_{\mathfrak{p} \in Spec(Z(\mathcal{A}))}$.

It implies that $Azu'_{\mathcal{A}}(\mathcal{K}) = \{\mathfrak{m} \in MaxSpec(Z(\mathcal{A})), \Lambda(\mathfrak{m}) \leq d\}$ and then, again by the upper semi-continuity of Λ, $Azu'_{\mathcal{A}}(\mathcal{K})$ is open. $\square$

4.2. **Reduced skein module.** Following [FTFKB25], for M a closed oriented 3-manifold, we consider a Heegaard splitting $M =: H_1 \underset{\Sigma}{\cup} H_2$ of M.

Then, the map $S_{-1}(\Sigma) \to S_{-1}(M)$ is surjective and we can consider $Spec(S_{-1}(M))$ as a subspace of $Spec(S_{-1}(\Sigma))$.

Also, since $\partial H_i = \Sigma$, we view $S_{\zeta}(H_i)$ as a $S_{\zeta}(\Sigma)$-module.

Moreover, by [FKBL19, Theorem 4.1], $Z(S_{\zeta}(\Sigma)) \simeq S_{-1}(\Sigma)$ through the map given by Theorem 2.2.

Proposition 4.5. [FKBL19, Theorem 5.1] *Let* $i \in \{1, 2\}$.
The algebra $\mathcal{A} := S_{\zeta}(\Sigma)$ *is almost Azumaya and* $\mathcal{K} := S_{\zeta}(H_i)$ *is a finitely generated* $\mathcal{A}$-*module.*

We now can talk about $Azu(S_{\zeta}(\Sigma))$ and $Azu'_{S_{\zeta}(\Sigma)}(S_{\zeta}(H_i))$. Let

$$D^2 := \dim_{Frac(Z(\mathcal{A}))}(\mathcal{A} \underset{Z(\mathcal{A})}{\otimes} Frac(Z(\mathcal{A})))$$

And

$$d = \dim_{Frac(Z(\mathcal{A}))}(\mathcal{K} \underset{Z(\mathcal{A})}{\otimes} Frac(Z(\mathcal{A})))$$

We will see in Proposition 4.7 that $D = d$.

4.3. **Non-central characters.** We are now ready to describe the important results about localized skein modules. The first result describes the localized skein modules of Σ at non-central characters.

Proposition 4.6. [GJS25, Theorem 1.1.4][KK25, Theorem 1.2] *For* $[\rho]$ *the character of a non-central representation of* $\chi(M)$, $\mathfrak{m}_{[\rho]} \in Azu(S_{\zeta}(\Sigma))$.

The second result describes the localized skein modules of the handlebodies H_i at non-central characters.

Proposition 4.7. [FKBL25, Theorem 12.1][KK25, Lemma 6.5] *For* $i \in \{1, 2\}$ *and* $[\rho]$ *the character of a non-central representation of* $\chi(M)$, $\mathfrak{m}_{[\rho]} \in Azu'_{S_{\zeta}(\Sigma)}(S_{\zeta}(H_i))$. *Moreover,* $D = d$.

These two results will be used in the following:

Theorem 4.8. [FTFKB25] *Let* $[\rho]$ *be the character of a non-central representation of* $S(M)$ *and let* $n_{[\rho]}$ *be the multiplicity of* $[\rho]$, *then:*

$$S_{\zeta}(M)_{[\rho]} \simeq S_{-1}(M)_{[\rho]} \simeq \mathbb{C}^{n_{[\rho]}}$$

Proof. Since the theorem of [FTFKB25] only addresses irreducible characters we explain how to use their proof in the general case.

The key idea is to notice that the only thing needed in [FTFKB25] about $[\rho]$ is to verify the hypothesis of [FTFKB25, Prop. 3.3] with both $(K, A) = (S_{-1}(\Sigma), S_\zeta(\Sigma))$ and $(K, A) = (S_{-1}(H_i), S_\zeta(H_i))$. In [FTFKB25], this is done in [FTFKB25, Prop. 5.4] by [FTFKB25, Prop. 4.2] and [FTFKB25, Theorem. 4.1].

However, Proposition 4.6 and Proposition 4.7 ensure that every non-central representation $[\rho]$ is in $(Azu(S_\zeta(\Sigma)) \bigcap_{i \in \{1,2\}} Azu'_{S_\zeta(\Sigma)}(S_\zeta(H_i)))$, and Proposition 4.3 and Proposition 4.4 shows that being in these sets are open conditions.

Therefore, [FTFKB25, Prop. 3.3] can be used with both $(K, A) = (S_{-1}(\Sigma), S_\zeta(\Sigma))$ and $(K, A) = (S_{-1}(H_i), S_\zeta(H_i))$ at the level of non-central characters. $\square$

4.4. **The total skein module.**

Since central characters are isolated and reduced when M is a rational homology sphere, in particular if $X(M)$ is finite, reduced skein modules at central characters are identical to localized skein modules. Then we have the following:

Proposition 4.9. [Kor25, Lemma 4.5] *Let M be an oriented closed 3-manifold with finite $X(M)$ and $[\rho], [\rho'] \in \chi(M)$ be two central characters, then $S_\zeta(M)_{[\rho]} \simeq S_\zeta(M)_{[\rho']}$*

For the sake of completeness, we provide the proof below.

Proof. Let $L, L' \in S_\zeta(M)$ be represented by links, let $K_1, \dots, K_n$ be the components of L, and let $r(L, L') = T_N(L) \sqcup L' - \prod_{i=1}^{n}(-tr(\rho(K_i)))L'$. Then, $S_{\zeta,[\rho]}(M)$ is the quotient of $S_\zeta(M)$ by all the possible relations of the form $r(L, L')$ (and likewise for $S_{\zeta,[\rho']}(M)$).

Using the fact that the skein relations are $H^1(M, \mathbb{Z}/2\mathbb{Z})$-homogeneous, for $\omega \in H^1(M, \mathbb{Z}/2\mathbb{Z})$, the automorphism $f_\omega : S_\zeta(M) \to S_\zeta(M)$ determined by $f_\omega(L) = (-1)^{\sum \omega(K_i)}L$, for L represented by link of components $K_1, \dots K_n$, is well defined.

Since ρ and ρ' are both central representations, there exists $\omega \in H^1(M, \mathbb{Z}/2\mathbb{Z})$ such that for any knot K, one has that $(-1)^{\omega(K)}tr(\rho(K)) = tr(\rho'(K))$.

Since $T_N(-X) = -T_N(X)$, the automorphism f_ω descends to an isomorphism $S_{\zeta,[\rho]}(M) \simeq S_{\zeta,[\rho']}(M)$. Using the fact that $[\rho]$ and $[\rho']$ are reduced completes the proof. $\square$

This is sufficient to prove Theorem 1.9:

Proof of Theorem 1.9. The right part of the decomposition is coming from Theorem 4.8 and the left part is from Proposition 4.9. $\square$

4.5. **Comparison with** [DKS25a].

Recall that $|X(M)|$ is the number of points of $X(M)$ counted with multiplicity. Let η be the count without multiplicity.

First we prove Remark 1.4:

In their work, [DKS25a] establishes two inequalities : $\eta \leq \dim_{\mathbb{Q}(A)} S(M)$ and $\dim_{\mathbb{Q}(A)} S(M) \leq \dim_{\mathbb{C}} \mathbb{C}[\chi(M)] = |X(M)|$. However, using Theorem 1.9, we have the inequality $|X(M)| \leq \dim_{\mathbb{Q}(A)} S(M)$, which gives their result without the first inequality and then without the reduced (equivalently $\eta = |X(M)|$) assumption.

It is worth noting that it is tempting to try to prove the inequality $\dim_{\mathbb{Q}(A)} S(M) \leq |X(M)|$ following the same path as [DKS25a] in our setting, through the decomposition of Section 3.2:

$$S(E_K(r), R_U) = F \bigoplus_i R_U/q_i^{s_i}$$

Where F is a free R_U-module.

However, here, -1 might be a root of U. In this case, we have $R_U \underset{A=-1}{\bigotimes} \mathbb{C} = 0$ and we cannot recover the dimension of $S_{-1}(E_K(r))$ by this decomposition.

References

[AM69]   Michael Francis Atiyah and I. G. MacDonald, *Introduction to commutative algebra/*, Addison-Wesley series in mathematics;, Addison-Wesley„ Reading, Mass.:, [1969].

[AM20]   Mohammed Abouzaid and Ciprian Manolescu, *A sheaf-theoretic model for* $\mathrm{SL}(2, \mathbb{C})$ *Floer homology*, J. Eur. Math. Soc. (JEMS) **22** (2020), no. 11, 3641–3695.

[BD25]   Giulio Belletti and Renaud Detcherry, *An effective proof of finiteness for kauffman bracket skein modules*, 2025, 2507.02589.

[BG02]   Ken A. Brown and Ken R. Goodearl, *Lectures on algebraic quantum groups*, Advanced Courses in Mathematics. CRM Barcelona, Birkhäuser Verlag, Basel, 2002.

[BKSW25]   Rhea Palak Bakshi, Seongjeong Kim, Shangjun Shi, and Xiao Wang, *On the Kauffman bracket skein module of* $(S^1 \times S^2) \# (S^1 \times S^2)$, J. Algebra **673** (2025), 103–137.

[Bul97]   Doug Bullock, *Rings of* $\mathrm{SL}_2(\mathbb{C})$*-characters and the Kauffman bracket skein module*, Comment. Math. Helv. **72** (1997), no. 4, 521–542.

[BW16]   Francis Bonahon and Helen Wong, *Representations of the Kauffman bracket skein algebra I: invariants and miraculous cancellations*, Invent. Math. **204** (2016), no. 1, 195–243.

[Det21]   Renaud Detcherry, *Infinite families of hyperbolic 3-manifolds with finite-dimensional skein modules*, J. Lond. Math. Soc. (2) **103** (2021), no. 4, 1363–1376.

[DKS25a]   Renaud Detcherry, Efstratia Kalfagianni, and Adam S. Sikora, *Kauffman bracket skein modules of small 3-manifolds*, Adv. Math. **467** (2025), Paper No. 110169, 45.

[DKS25b]   ———, *Skein modules and character varieties of Seifert manifolds*, Trans. Amer. Math. Soc. Ser. B **12** (2025), 1264–1288.

[DW21]   Renaud Detcherry and Maxime Wolff, *A basis for the Kauffman skein module of the product of a surface and a circle*, Algebr. Geom. Topol. **21** (2021), no. 6, 2959–2993.

[FG00]   Charles Frohman and Răzvan Gelca, *Skein modules and the noncommutative torus*, Trans. Amer. Math. Soc. **352** (2000), no. 10, 4877–4888.

[FKBL19]   Charles Frohman, Joanna Kania-Bartoszynska, and Thang Lê, *Unicity for representations of the Kauffman bracket skein algebra*, Invent. Math. **215** (2019), no. 2, 609–650.

[FKBL25]   Charles D. Frohman, Joanna Kania-Bartoszynska, and Thang T. Q. Lê, *Sliced skein algebras and geometric Kauffman bracket*, Adv. Math. **463** (2025), Paper No. 110118, 65.

[FTFKB25]   Mohammad Farajzadeh-Tehrani, Charles Frohman, and Joanna Kania-Bartoszynska, *The Kauffman bracket skein module at an irreducible representation*, Quantum topology (2025).

[GJS23]   Sam Gunningham, David Jordan, and Pavel Safronov, *The finiteness conjecture for skein modules*, Invent. Math. **232** (2023), no. 1, 301–363.

[GJS25]   Iordan Ganev, David Jordan, and Pavel Safronov, *The quantum Frobenius for character varieties and multiplicative quiver varieties*, J. Eur. Math. Soc. (JEMS) **27** (2025), no. 7, 3023–3084.

[GM19]   Patrick M. Gilmer and Gregor Masbaum, *On the skein module of the product of a surface and a circle*, Proc. Amer. Math. Soc. **147** (2019), no. 9, 4091–4106.

[GS24]   Sam Gunningham and Pavel Safronov, *Deformation quantization and perverse sheaves*, 2024, 2312.07595.

[Har77]   Robin Hartshorne, *Algebraic geometry*, Graduate Texts in Mathematics, vol. No. 52, Springer-Verlag, New York-Heidelberg, 1977.

[HP93]   Jim Hoste and Józef H. Przytycki, *The* $(2, \infty)$*-skein module of lens spaces; a generalization of the Jones polynomial*, J. Knot Theory Ramifications **2** (1993), no. 3, 321–333.

[HP95]   ———, *The Kauffman bracket skein module of* $S^1 \times S^2$, Math. Z. **220** (1995), no. 1, 65–73.

[Hun80]   Thomas W. Hungerford, *Algebra*, Graduate Texts in Mathematics, vol. 73, Springer-Verlag, New York-Berlin, 1980, Reprint of the 1974 original.

[KK25]   Hiroaki Karuo and Julien Korinman, *Azumaya loci of skein algebras*, 2025, 2211.13700.

[Kor25]   Julien Korinman, *Skein modules of closed 3 manifolds define line bundles over character varieties*, 2025, 2501.02617.

[Lê06]   Thang T. Q. Lê, *The colored Jones polynomial and the A-polynomial of knots*, Adv. Math. **207** (2006), no. 2, 782–804.

[Lê15]   ———, *On Kauffman bracket skein modules at roots of unity*, Algebr. Geom. Topol. **15** (2015), no. 2, 1093–1117.

[Mro11]   Maciej Mroczkowski, *Kauffman bracket skein module of the connected sum of two projective spaces*, J. Knot Theory Ramifications **20** (2011), no. 5, 651–675.

[Prz91]   Józef H. Przytycki, *Skein modules of 3-manifolds*, Bull. Polish Acad. Sci. Math. **39** (1991), no. 1-2, 91–100.

[PS00]   Józef H. Przytycki and Adam S. Sikora, *On skein algebras and* $Sl_2(\mathbf{C})$*-character varieties*, Topology **39** (2000), no. 1, 115–148.

[Tur88]   V. G. Turaev, *The Conway and Kauffman modules of a solid torus*, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **167** (1988), 79–89, 190.

Institut de Mathématiques de Bourgogne, UMR 5584 CNRS, Université de Bourgogne, F-21000, Dijon, France

Email address: edwin.kitaeff@ube.fr